Name:………………………………………………………………………………….

Student Number: ………………………………………………………………..

Test 4 on WPPH16001.2017-2018
“Electricity and Magnetism”

Content: 10 pages (including this cover page)

Friday May 4 2018; A. Jacobshal 01, 9:00-11:00

- Write your full name and student number
- Write your answers in the designated area
- Reversed sides of each page are left blank intentionally and could be used for draft answers
- Read the questions carefully
- Please, do not use a pencil
- Books, notes, phones, and tablets are not allowed. Calculators and dictionaries are allowed.

Exam drafted by (name first examiner) Maxim S. Pchenitchnikov
Exam reviewed by (name second examiner) Steven Hoekstra

The weighting of the questions and the grading scheme (total number of points, and mention the number of points at each (sub)question).
Grade = 1 + 9 x (score/max score).

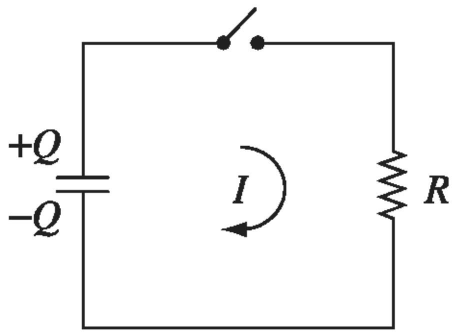
For administrative purposes; do NOT fill the table

	Maximum points	Points scored
Question 1	10	
Question 2	13	
Question 3	10	
Question 4	7	
Total	**40**	

Final mark: _________

Question 1. (10 points)

A capacitor C has been charged up to potential V_0; at time $t = 0$, it is connected to a resistor R (the switch is flipped on), and begins to discharge (see Figure).



1. Determine the charge on the capacitor as a function of time, $Q(t)$. (5 points)

2. What is the current through the resistor, $I(t)$? (1 point)

3. By integrating the Joule heating law, confirm that the heat delivered to the resistor is equal to the energy lost by the capacitor. (3 points)

4. What was the original energy stored in the capacitor? (1 point)

Model answers (Griffiths, Problem 7.2a,b)

1. $V = Q/C = IR.$ (1 point)

Because positive I means the charge on the capacitor is decreasing:

$$\frac{dQ}{dt} = -I = -\frac{1}{RC}Q$$ (1 point)

so $Q(t) = Q_0 e^{-t/RC}$ (1 point)

$$Q_0 = Q(0) = CV_0.$$ (1 point)

so $\boxed{Q(t) = CV_0 e^{-t/RC}.}$ (1 point)

2. $I(t) = -\frac{dQ}{dt}$

$$= CV_0 \frac{1}{RC} e^{-t/RC} = \boxed{\frac{V_0}{R} e^{-t/RC}.}$$ (1 point)

3. The energy delivered to the resistor is

$$\int_0^\infty P\,dt = \int_0^\infty I^2 R\,dt = \frac{V_0^2}{R}\int_0^\infty e^{-2t/RC}dt =$$ (1 point)

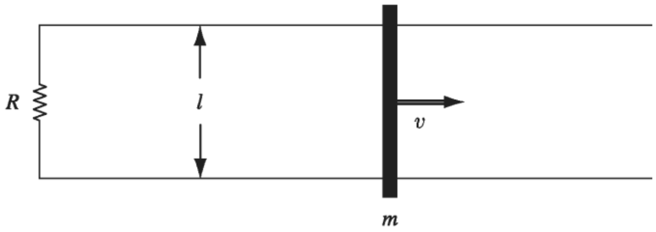
$$\frac{V_0^2}{R}\left(-\frac{RC}{2}e^{-2t/RC}\right)\Big|_0^\infty$$ (1 point)

$$= \frac{1}{2}CV_0^2$$ (1 point)

4. $W = \boxed{\frac{1}{2}CV_0^2.}$ (of course because energy is conserved!) (1 point)

Question 2. (13 points)

A metal bar of mass m slides frictionlessly on two parallel conducting rails a distance l apart; the whole set is placed on a horizontal plane (See Figure). A resistor R is connected across the rails, and a uniform magnetic field **B**, pointing into the page, fills the entire region.



1. If the bar moves to the right at speed v, what is the current in the resistor? In what direction does it flow? (2 points)
2. What is the magnetic force on the bar? In what direction? (3 points)
3. If the bar starts out with speed v_0 at time $t = 0$, and is left to slide, what is its speed at a later time t? (3 points)
4. Find the energy delivered to the resistor (Tip: make a substitution $\alpha \equiv \frac{B^2 l^2}{mR}$) (4 points)
5. Does the answer make any sense? Why? (1 point)

Model answers (Problem 7.7):

1. $\mathcal{E} = -\frac{d\Phi}{dt} = -Bl\frac{dx}{dt} = -Blv;\ \mathcal{E} = IR \Rightarrow \boxed{I = \frac{Blv}{R}.}$ (1 point)
(never mind the minus sign)
Current flows anticlockwise (downward through the resistor) (1 point)

2. $F = quB = B\int Idt\frac{dl}{dt} = BIl$ (u is the charge speed along l) (1 point)
$F = \frac{B^2 l^2 v}{R}$ (1 point)
To the **left** direction (according to Lenz) (1 point)

3. $F = ma = m\frac{dv}{dt} = -\frac{B^2 l^2}{R}v$ (1 point)
$\frac{dv}{dt} = -(\frac{B^2 l^2}{Rm})v \Rightarrow \boxed{v = v_0 e^{-\frac{B^2 l^2}{mR}t}.}$ (2 points)

4. $\frac{dW}{dt} = I^2 R$ (1 point)
$= \frac{B^2 l^2 v^2}{R^2}R = \frac{B^2 l^2}{R}v_0^2 e^{-2\alpha t}$, where $\alpha \equiv \frac{B^2 l^2}{mR}$; $\frac{dW}{dt} = \alpha m v_0^2 e^{-2\alpha t}$. (1 point)
$W = \alpha m v_0^2 \int_0^\infty e^{-2\alpha t}dt = \alpha m v_0^2 \left.\frac{e^{-2\alpha t}}{-2\alpha}\right|_0^\infty = \alpha m v_0^2 \frac{1}{2\alpha} = \frac{1}{2}m v_0^2.$ (2 points)

5. $\frac{1}{2}m v_0^2$. is the initial kinetic energy of the bar; it all has spent on heat – so that the law of energy conservation works! (1 point)

Question 3. (10 points)

A long solenoid with radius a and n turns per unit length carries a time-dependent current $I(t)$ in the direction $\hat{\boldsymbol{\varphi}}$. Find the electric field **E** (**magnitude and direction**) in the quasistatic approximation (i.e. the magnetic field can be considered as produced by a constant current):

1. At a distance s from the axis inside the solenoid (4 points)
2. At a distance s from the axis outside the solenoid (3 points)
3. Find the self-inductance $\mathcal{L}$ per unit length of such a solenoid (3 points)

Tip: the magnetic field of the solenoid is $\mathbf{B} = \begin{cases} \mu_0 n I\,\hat{\mathbf{z}}, & (s < a) \\ \mathbf{0}, & (s > a) \end{cases}$

Model answers (Problem 7.15, 7.24):

1. Faraday's law $\oint_{\mathcal{P}} \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\boldsymbol{a}$

For an "amperian loop" of radius $s < a$,

$\Phi = B\pi s^2 = \mu_0 n I \pi s^2$ (1 point)

$\oint \mathbf{E}\cdot d\mathbf{l} = E\,2\pi s = -\frac{d\Phi}{dt} = -\mu_0 n\pi s^2 \frac{dI}{dt}$ (2 points)

$\boxed{\mathbf{E} = -\frac{\mu_0 n s}{2}\frac{dI}{dt}\,\hat{\phi}.}$ (1 point)

2. For an "amperian loop" of radius $s > a$:

$\Phi = B\pi a^2 = \mu_0 n I \pi a^2$ (1 point)

$E\,2\pi s = -\mu_0 n \pi a^2 \frac{dI}{dt}$ (1 point)

$\boxed{\mathbf{E} = -\frac{\mu_0 n a^2}{2s}\frac{dI}{dt}\,\hat{\phi}.}$ (1 point)

3. Flux through a single turn $\Phi_1 = \mu_0 n I \pi a^2$ (1 point)

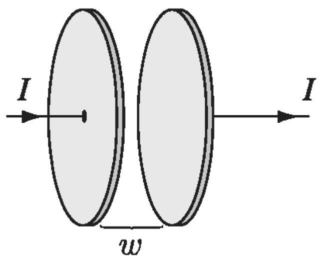
In a length l there are nl such turns, so the total flux is $\Phi = \mu_0 n^2 \pi a^2 l\, I$ (1 point)

The self-inductance is given by $\Phi = L\,I$

so the self-inductance per unit length is $\mathcal{L} = \frac{L}{l} = \mu_0 n^2 \pi a^2$ (1 point)

Question 4. (7 points)

The constant current I runs through thin wires that connect to the centers of the plates that form a parallel-plate capacitor (see Figure). The radius of the capacitor is a, and the separation of the plates is $w<<a$. Assume that the current flows out over the plates in such a way that the surface charge is uniform, at any given time, and is zero at $t = 0$.



1. Find the electric field between the plates, as a function of t. (3 points)

2. Find the displacement current $I_d = J_d A$ (where A is the area) through a circle of radius s in the plane midway between the plates. (1 point)

3. Using this circle as your "Amperian loop," and the flat surface that spans it, find the magnetic field at a distance s from the axis. (3 points)

__**Model answers (Problem 7.35/7.34; also considered at lectures):**__

1. $\mathbf{E} = \frac{\sigma(t)}{\epsilon_0}\hat{\mathbf{z}};$ (1 point)

$$\sigma(t) = \frac{Q(t)}{\pi a^2} = \frac{It}{\pi a^2}$$ (1 point)

$$\mathbf{E} = \boxed{\frac{It}{\pi\epsilon_0 a^2}\hat{\mathbf{z}}.}$$ (1 point)

2. $I_{d_{\text{enc}}} = J_d \pi s^2 = \epsilon_0 \frac{dE}{dt}\pi s^2 = \boxed{I\frac{s^2}{a^2}.}$ (1 point)

3. $\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0\epsilon_0 \int \frac{\partial \mathbf{E}}{\partial t}\cdot d\boldsymbol{a}$ or $\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{d_{\text{enc}}}$ (1 point)

$$B\,2\pi s = \mu_0 I \frac{s^2}{a^2}$$ (1 point)

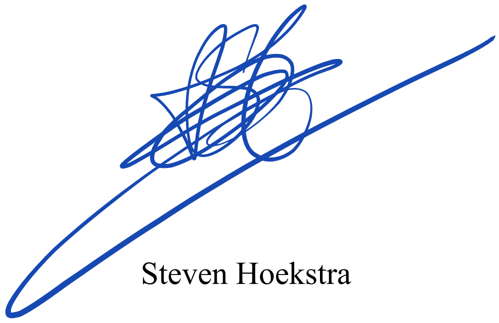
$$\mathbf{B} = \frac{\mu_0 I}{2\pi a^2} s\,\hat{\boldsymbol{\phi}}$$ (1 point)

Alternatively, in the cylindrical system of coordinates:

$$\frac{1}{s}\left[\frac{\partial}{\partial s}(sB_\varphi)\right]\hat{\mathbf{z}} = \mu_0\epsilon_0 \frac{\partial \mathbf{E}_z}{\partial t}$$ (1 point)

$$\frac{\partial}{\partial s}(sB_\varphi) = \frac{\mu_0\epsilon_0 I}{\epsilon_0 \pi a^2} s$$ (1 point)

$$sB_\varphi = \frac{\mu_0 \beta}{\pi a^2}\frac{s^2}{2}; \mathbf{B} = \frac{\mu_0 I}{2\pi a^2} s\hat{\boldsymbol{\phi}}$$ (1 point)

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May 1 2018